



6.5 Linear dependence and independence of a collection of vectors





Introduction

- In this topic, we will
 - Define linear dependence and independence of a given vector $\mathbf{v}$ on a collection of vectors
 - Determine how to check if a vector linearly depends on others
 - Define the rank of a matrix
 - Define a linearly dependent collection of vectors,
and a linearly independent collection of vectors
 - Determine how to check if a collection of vectors is linearly dependent or independent
 - Consider a few results





Definition

- A vector $\mathbf{u}$ is said to be *linearly dependent* on a collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ if $\mathbf{u}$ can be written as a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$

- That is, $\mathbf{u}$ is linearly dependent on $\mathbf{v}_1, \dots, \mathbf{v}_n$ if the problem

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}$$

has at least one solution

- If $\mathbf{u}$ is not linearly dependent on a collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, we say $\mathbf{u}$ is *linearly independent* of $\mathbf{v}_1, \dots, \mathbf{v}_n$

- That is, the problem

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}$$

has no solutions





The zero vector

- Note that the vector $\mathbf{0}_m$ is *linearly dependent* on any collection of n m -dimensional vectors as

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}_m$$

always has at least one solution:

$$\alpha_1 = \cdots = \alpha_n = 0$$





Definition

- Thus, saying that a vector $\mathbf{u}$ is linearly dependent on a collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is the same as saying that the system of linear equations corresponding to

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}$$

has at least one solution

- Similarly, saying that a vector $\mathbf{u}$ is linearly independent of a collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is the same as saying that the system of linear equations corresponding to

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}$$

has no solution





Dependence of vectors

Theorem

If $\mathbf{u}$ is a non-zero vector and $\mathbf{u}$ linearly depends on a collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, then there must be at least one vector $\mathbf{v}_k$ that linearly depends on

$\mathbf{u}, \mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$

Proof:

If $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}$ and $\mathbf{u}$ is not the zero vector, then the solution vector α cannot be the zero vector.

Thus, at least one entry $\alpha_k \neq 0$.

Thus, $\alpha_k \mathbf{v}_k = \mathbf{u} - \alpha_1 \mathbf{v}_1 - \dots - \alpha_{k-1} \mathbf{v}_{k-1} - \alpha_{k+1} \mathbf{v}_{k+1} - \dots - \alpha_n \mathbf{v}_n$

$$\mathbf{v}_k = \frac{1}{\alpha_k} \mathbf{u} - \frac{\alpha_1}{\alpha_k} \mathbf{v}_1 - \dots - \frac{\alpha_{k-1}}{\alpha_k} \mathbf{v}_{k-1} - \frac{\alpha_{k+1}}{\alpha_k} \mathbf{v}_{k+1} - \dots - \frac{\alpha_n}{\alpha_k} \mathbf{v}_n \quad \blacksquare$$





Dependence of vectors

- For example, given

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

the corresponding system of linear equations has one unique solution $\alpha = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

- Thus
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.2 \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$





Definitions

- Definition:

Recall that the rank of an augmented matrix $(A:\mathbf{b})$ is the number of non-zero rows any row-equivalent augmented matrix that is in row-echelon form

- Note that we can perform row operations on simply a matrix A without the augmentation with a target vector $\mathbf{b}$

- Definition

The rank of a matrix A is the number of non-zero rows of any row-equivalent matrix that is in row-echelon form





Terminology

- For example, because

$$A = \begin{pmatrix} 1 & 1.4 & -9.6 \\ 5 & 7 & -3 \\ -0.5 & -0.7 & 3 \end{pmatrix} \sim \begin{pmatrix} 5 & 7 & -3 \\ 0 & 0 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

thus, the $\text{rank}(A) = 2$

- For example, because

$$B = \begin{pmatrix} 3.6 & 3.9 & -5.1 & 4.2 \\ -2.4 & -0.9 & 3.7 & 6.4 \\ 4 & 3 & -1 & 0 \\ -2.8 & 0.9 & 2.7 & 3 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & -1 & 0 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

thus, the $\text{rank}(B) = 4$





Terminology

- Finally, because

$$C = \begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 0 & 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

thus, the $\text{rank}(C) = 2$





Approach to determining dependence

- If all you are asked “Is $\mathbf{u}$ linearly dependent on $\mathbf{v}_1, \dots, \mathbf{v}_n$?” then all you must do is check if

$$\text{rank}(V) = \text{rank}(V : \mathbf{u})$$

where

$$V = (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n)$$





Examples

- For example, $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

depends on

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

because

$$(V : \mathbf{u}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \end{array} \right)$$

and $\text{rank}(V) = 3 = \text{rank}(V : \mathbf{u})$ for all $\mathbf{u}$





Examples

- For example, $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

depends on

$$\mathbf{v}_1 = \begin{pmatrix} \delta_1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \alpha \\ \delta_2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} \beta \\ \gamma \\ \delta_3 \end{pmatrix}$$

where $\delta_1 \neq 0$, $\delta_2 \neq 0$ and $\delta_3 \neq 0$ because

$$(V : \mathbf{u}) = \left(\begin{array}{ccc|c} \delta_1 & \alpha & \beta & u_1 \\ 0 & \delta_2 & \gamma & u_2 \\ 0 & 0 & \delta_3 & u_3 \end{array} \right)$$

and $\text{rank}(V) = 3 = \text{rank}(V : \mathbf{u})$ for all $\mathbf{u}$





Examples

- For example, $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

depends on

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

because (without partial pivoting):

$$(V : \mathbf{u}) = \left(\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 2 & 5 & 8 & 1 \\ 3 & 6 & 9 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

and $\text{rank}(V) = 2 = \text{rank}(V : \mathbf{u})$ for this $\mathbf{u}$





Examples

- For example, $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

does not depends on

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

because (without partial pivoting):

$$(V:\mathbf{u}) = \left(\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 2 & 5 & 8 & 1 \\ 3 & 6 & 9 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

and $\text{rank}(V) = 2 \neq 3 = \text{rank}(V:\mathbf{u})$ for this $\mathbf{u}$





Examples

- For example, $\mathbf{u} = \begin{pmatrix} 4.35 \\ -3.5 \\ 1.9 \end{pmatrix}$

does not depend on

$$\mathbf{v}_1 = \begin{pmatrix} 4.9 \\ 7 \\ 2.8 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 9 \\ 10 \\ 5 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -13.2 \\ -6 \\ -6.9 \end{pmatrix}$$

because

$$(V : \mathbf{u}) = \left(\begin{array}{ccc|c} 4.9 & 9 & -13.2 & 4.35 \\ 7 & 10 & -6 & -3.5 \\ 2.8 & 5 & -6.9 & 1.9 \end{array} \right) \sim \left(\begin{array}{ccc|c} 7 & 10 & -6 & 3.5 \\ 0 & 2 & -9 & 6.8 \\ 0 & 0 & 0 & -0.1 \end{array} \right)$$

and $\text{rank}(V) = 2 \neq 3 = \text{rank}(V : \mathbf{u})$ for this $\mathbf{u}$





Definitions

Definition

A collection of vectors is *linearly independent* if no vector depends on the other vectors in the collection.

- That is, no vector in collection set can be written as a linear combination of the other vectors

Definition

A collection of vectors is *linearly dependent* if it is not linearly independent.

- That is, there is a vector in the collection that can be written as a linear combination of the other vectors
- That is, there is at least one vector in the collection that depends on the others





Collections with the zero vector

Definition

Note that the collection containing only the zero vector is linearly dependent, because the zero vector is the sum of no vectors:

- Recall that the sum of no numbers is equal to 0
- Consequently, a collection with only one non-zero vector $\mathbf{v}$ is always guaranteed to be linearly independent





Collections with the zero vector

Theorem

Any collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of which one of them is the zero vector is linearly dependent.

Proof:

Assume $\mathbf{v}_k$ is the zero vector.

Thus $\mathbf{v}_k = 0 \cdot \mathbf{v}_1 + \dots + 0 \cdot \mathbf{v}_{k-1} + 0 \cdot \mathbf{v}_{k+1} + \dots + 0 \cdot \mathbf{v}_n$

Thus, one vector depends on the others.

Thus, the collection is linearly dependent. ■





Determining linear independence

Theorem

A collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent if and only if $\text{rank}(V) = n$ where $V = (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n)$.

Proof:

Let us prove this by induction.

If $n = 1$, then a single non-zero vector is linearly independent, and a single zero vector is linearly dependent.

If $V = (\mathbf{v}_1)$, then if $\mathbf{v}_1$ is not the zero vector, $\text{rank}(V) = 1 = n$, and if $\mathbf{v}_1$ is the zero vector, $\text{rank}(V) = 0 \neq n$.





Determining linear independence

Next, assume for an arbitrary n that a collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent if and only if $\text{rank}(V) = n$.

For $n + 1$, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ are not linearly independent, then adding another vector $\mathbf{v}_{n+1}$ into the collection cannot make the collection linearly independent.

- If $\mathbf{v}_k = \alpha_1 \mathbf{v}_1 + \dots + \alpha_{k-1} \mathbf{v}_{k-1} + \alpha_{k+1} \mathbf{v}_{k+1} + \dots + \alpha_n \mathbf{v}_n$
then $\mathbf{v}_k = \alpha_1 \mathbf{v}_1 + \dots + \alpha_{k-1} \mathbf{v}_{k-1} + \alpha_{k+1} \mathbf{v}_{k+1} + \dots + \alpha_n \mathbf{v}_n + 0 \mathbf{v}_{n+1}$

Also, as $\text{rank}(V) < n$, so $\text{rank}(\mathbf{v}_1 \cdots \mathbf{v}_n \mathbf{v}_{n+1}) < n + 1$.





Determining linear independence

For $n + 1$, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent then by assumption, $\text{rank}(V) = n$.

Suppose we add another vector $\mathbf{v}_{n+1}$ into the collection.

Now, $\text{rank}(V : \mathbf{v}_{n+1})$ must equal either n or $n + 1$.

If $\text{rank}(V : \mathbf{v}_{n+1}) = n = \text{rank}(V)$, then there is a unique solution to $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{v}_{n+1}$, so $\mathbf{v}_{n+1}$ depends on the first n vectors, and $\text{rank}(\mathbf{v}_1 \cdots \mathbf{v}_n \mathbf{v}_{n+1}) = n \neq n + 1$.

If $\text{rank}(V : \mathbf{v}_{n+1}) = n + 1$, then there is no solution to $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{v}_{n+1}$, so $\mathbf{v}_{n+1}$ is independent of the first n vectors.

Thus, the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}$ are linearly independent and $\text{rank}(\mathbf{v}_1 \cdots \mathbf{v}_n \mathbf{v}_{n+1}) = n + 1$. ■





Determining linear independence

Theorem

A collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbf{R}^m$ where $m < n$ is never linearly independent.

Proof:

The maximum rank of an $m \times n$ matrix V is $\min\{m, n\}$,
and if $m < n$ then $\text{rank}(V) < n$

Thus, it is impossible for $\text{rank}(V) = n$. ■





Determining linear independence

Theorem

A collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbf{R}^m$ is linearly independent if and only if the only solution to

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

is the trivial solution.

Proof:

The matrix V is an $m \times n$ matrix.

Consider solving $(V : \mathbf{0}_m)$.

If $\text{rank}(V : \mathbf{0}) = n$, then there is a unique solution,

and that unique solution is $\alpha_1 = \dots = \alpha_n = 0$.

If $\text{rank}(V : \mathbf{0}) < n$, then there is at least one free variable,

and so there are infinitely many solutions. ■





Question on linear dependence

Theorem

Given a collection of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbf{R}^m$ that is known to be linearly dependent, then there is a first vector $\mathbf{v}_k$ that linearly depends on the first $k - 1$ linearly independent vectors.

Proof:

The matrix V must have $\text{rank}(V) < n$.

Thus, there is at least one free variable if we were solving $(V : \mathbf{0})$.

Find a row-equivalent matrix in row-echelon form.

From our algorithm, there is a first column that does not contain a leading non-zero entry.

If it is the first column, that column must be the zero vector,
which is not linearly independent.





Question on linear dependence

Otherwise, there is another first column that does not contain a leading non-zero entry: let that be Column k .

$$V \sim \begin{pmatrix} \square & ? & ? & ? & ? & ? & ? \\ 0 & \square & ? & ? & ? & ? & ? \\ 0 & 0 & \square & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & \square & ? & ? \\ 0 & 0 & 0 & 0 & 0 & \square & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \downarrow \\ \text{Column } k \end{matrix} \quad \begin{pmatrix} \square & ? & ? \\ 0 & \square & ? \\ 0 & 0 & \square \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \square & ? & ? & ? \\ 0 & \square & ? & ? \\ 0 & 0 & \square & ? \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, given the first $k-1$ vectors,

the matrix $(\mathbf{v}_1 \cdots \mathbf{v}_{k-1})$ must have rank $k-1$.

Thus, the augmented matrix $(\mathbf{v}_1 \cdots \mathbf{v}_{k-1} : \mathbf{v}_k)$ must also have rank $k-1$.

Therefore, there is a unique solution for $\mathbf{v}_k$ as a linear combination of the first $k-1$ linearly independent vectors. ■





Summary

- Following this topic, you now
 - Understand when and how to determine if a vector depends on a given collection of vectors
 - Know the definition of the rank of a matrix
 - Understand what is a linearly dependent and what is a linearly independent set of vectors
 - Understand that a collection of n vectors is linearly independent if and only the rank of the corresponding matrix is also n
 - Are aware that, for example 5 vectors in $\mathbf{R}^4$ can never be linearly independent
 - Are aware that, given n linearly dependent vectors, we can find a first vector that linearly depends on the previous linearly independent vectors





References

- [1] https://en.wikipedia.org/wiki/Linear_independence





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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